

Lesson 26. Local Minima and Maxima, cont.

Practice!

Use what we learned in Lesson 25 to solve the following problems.

Problem 1. Find the local minimum and maximum values and saddle points of $f(x, y) = 2xy - 4x - 2y - 2x^2 - y^2$.

Problem 2. Find the local minimum and maximum values and saddle points of $f(x, y) = 2 - x^4 + 2x^2 - y^2$.

Problem 3. Find the local minimum and maximum values and saddle points of $f(x, y) = x^3 - 3x + 3xy^2$.

Problem 1. Find the local minimum and maximum values and saddle points of $f(x,y) = 2xy - 4x - 2y - 2x^2 - y^2$.

$f_x(x,y) = 2y - 4 - 4x$ $\qquad$ $f_y(x,y) = 2x - 2 - 2y$

Find critical points:

$$\left.\begin{array}{l} -4x + 2y = 4 \quad (1) \\ 2x - 2y = 2 \quad (2) \end{array}\right\} \quad (2) \Rightarrow x = y + 1 \ (3) \xrightarrow{\text{sub into } (1)} -4y - 4 + 2y = 4$$

$$\Rightarrow -2y = 8 \Rightarrow y = -4$$

$$\xrightarrow{\text{sub into } (3)} x = -4 + 1 = -3$$

$\Rightarrow$ Critical points: $(-3,-4)$

Second derivative test:

$f_{xx}(x,y) = -4$ $\qquad$ $f_{yy}(x,y) = -2$ $\qquad$ $f_{xy}(x,y) = 2$.

$$\Rightarrow D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}(x,y)^2$$
$$= (-4)(-2) - 4 = 4$$

$(-3,-4)$: $\quad D(-3,-4) = 4$ $\quad \Rightarrow f(-3,-4) = 10$ is a local maximum

$f_{xx}(-3,-4) = -4$

Problem 2. Find the local minimum and maximum values and saddle points of $f(x, y) = 2 - x^4 + 2x^2 - y^2$.

$f_x(x,y) = -4x^3 + 4x$ $\qquad$ $f_y(x,y) = -2y$

Find critical points:

$$\left.\begin{array}{r} -4x^3 + 4x = 0 \quad ① \\ -2y = 0 \quad ② \end{array}\right\} \quad ① \Rightarrow -4x(x^2-1) = 0 \qquad ② \Rightarrow y = 0$$

$$\Rightarrow -4x(x+1)(x-1) = 0$$

$$\Rightarrow x = 0, 1, -1$$

$\Rightarrow$ Critical points: $(0,0)$, $(1,0)$, $(-1,0)$

Second derivative test:

$f_{xx}(x,y) = -12x^2 + 4$ $\qquad$ $f_{yy}(x,y) = -2$ $\qquad$ $f_{xy}(x,y) = 0$

$$\Rightarrow D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}(x,y)^2$$
$$= 24x^2 - 8$$

$(0,0)$: $\quad D(0,0) = -8 \quad \Rightarrow \quad (0,0)$ is a saddle point

$(1,0)$: $\quad \left.\begin{array}{l} D(1,0) = 16 \\ f_{xx}(1,0) = -8 \end{array}\right\} \Rightarrow f(1,0) = 3$ is a local maximum

$(-1,0)$: $\quad \left.\begin{array}{l} D(-1,0) = 16 \\ f_{xx}(-1,0) = -8 \end{array}\right\} \Rightarrow f(-1,0) = 3$ is a local maximum

Problem 3. Find the local minimum and maximum values and saddle points of $f(x, y) = x^3 - 3x + 3xy^2$.

$f_x(x,y) = 3x^2 - 3 + 3y^2$ $\qquad$ $f_y(x,y) = 6xy$

Find critical points:

$$\left.\begin{array}{r} 3x^2 + 3y^2 = 3 \quad (1) \\ 6xy = 0 \quad (2) \end{array}\right\} \quad (2) \Rightarrow x = 0 \text{ or } y = 0$$

If $x = 0$: $(1) \Rightarrow y^2 = 1 \Rightarrow y = 1$ or -1

If $y = 0$: $(1) \Rightarrow x^2 = 1 \Rightarrow x = 1$ or -1

$\Rightarrow$ Critical points: $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$

Second derivative test:

$f_{xx}(x,y) = 6x$ $\qquad$ $f_{yy}(x,y) = 6x$ $\qquad$ $f_{xy}(x,y) = 6y$

$$\Rightarrow D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - f_{xy}(x,y)^2$$
$$= 36x^2 - 36y^2$$

$(0,1)$: $D(0,1) = -36 \Rightarrow (0,1)$ is a saddle point

$(0,-1)$: $D(0,-1) = -36 \Rightarrow (0,-1)$ is a saddle point

$(1,0)$: $\left.\begin{array}{l} D(1,0) = 36 \\ f_{xx}(1,0) = 6 \end{array}\right\} \Rightarrow f(1,0) = -2$ is a local minimum

$(-1,0)$: $\left.\begin{array}{l} D(-1,0) = 36 \\ f_{xx}(-1,0) = -6 \end{array}\right\} \Rightarrow f(-1,0) = 2$ is a local maximum